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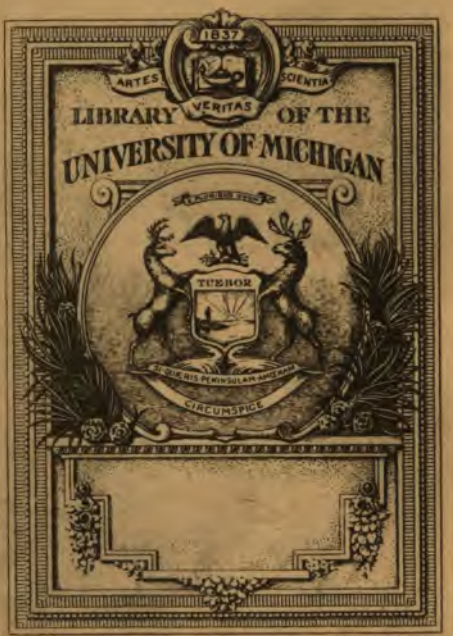

THE GIFT OF
Prof. L. C. Karpinski

METHOD IN GEOMETRY

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BY

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PREFACE.

THE author offers in this little monograph a brief outline of the method used in his class room. While many teachers no doubt have followed much the same course, it is offered with the hope that to some young teacher it may be a suggestion that will lead to more effective teaching. The monograph is a revision of a paper read before the Michigan Schoolmasters' Club, in April, 1902.

METHOD IN GEOMETRY.

Early Geometry.

The oldest traces of geometry are found among the early Egyptians and Babylonians. Their knowledge was, however, empirical and made simply to serve practical purposes. Many of their rules were simple approximations; the ratio of the circumference of a circle to its diameter was known to be a little greater than 3, but in practical measurements the circumference was taken to be three times the diameter.

A recent discovery of the papyrus written by Ahmes perhaps as early as 2000 B.C., gives us the earliest knowledge that we have of Egyptian geometry. There is no attempt in the writings of Ahmes at a science of the subject; there are no theorems or even general lines of procedure — the subject consists of a few special rules discovered by experimenting and induction. Neither were all the rules accurate; for example, the area of an isosceles triangle was found by taking half the product of the base by one of the equal sides.

The geometry of the Egyptians and Babylonians, such as it was, was sought after and studied by the

philosophers of Greece, and by the Greeks finally worked into a science. Thales (640 B.C. to 548 B.C.) and his pupil Pythagoras traveled and studied in Egypt and gave a valuable contribution to the subject of geometry, the latter giving us the theorem known by his name; but it remained for Euclid, about 300 B.C., to give to the world the science of geometry. For clearness of thought, exactness of truths, and excellence of logic Euclid is a model. With but slight change it has been used as a text-book for twenty-two hundred years. In England to-day his work holds almost universal sway.

Just how much of Euclid's work was his own and how much was compiled from former discoveries, is uncertain. It is known, however, that geometry did not come from the mind of Euclid "almost as perfect as Minerva from the head of Jove," but that a great deal of his material was drawn from Thales, Pythagoras and his school, and from Eudoxus. What Euclid did was to systematize the known knowledge of geometry and from it to discover new; that is, he began with definite ideas and self-evident truths, and, through a process of deduction, established the truth of such theorems as were handed down to him, and discovered some new ones. Whether he gave us much that was new or not, for the work he did he deserves to be ranked with the world's great educators and philosophers. "Geometry is the perfection of logic, Euclid is as classic as Homer."

The Educational Value of Geometry.

Most studies are taught for two purposes: (1) for the facts which may be of practical value in daily life, or that one needs in order to be really intelligent; (2) for the mental discipline to be obtained from acquiring these facts. In the broad sense this is true of geometry; the practical side of geometry, however, *i.e.* the measurement of surfaces and volumes, is usually taken up in arithmetic under the head of Mensuration, before the formal study of geometry is begun.

The study, then, of deductive geometry is almost entirely a disciplinary study, a lesson in logic. While each new fact discovered must be remembered as a basis for the discovery of other facts, yet geometry should be studied chiefly not for the facts it teaches but for the discipline it affords in apprehending the relations existing between these facts. When Euclid's followers were criticised for teaching that "any two sides of a triangle are together greater than the third side," as teaching that which even the beasts of the field know, the reply was, "We are not seeking to teach facts as much as the power to discover facts." Euclid recognized the truth that knowledge for its own sake is worth while, even if it cannot be used for practical ends. The story goes that he was once asked by a pupil who had just begun the study of geometry, "What do you gain by learning all this?" Euclid ordered

his servant to give him some coppers, "since he must have gain out of what he learns."

Suggestions on teaching Geometry.

Since the study of geometry is primarily a lesson in logic, it follows that the demonstrated propositions are not put there simply to be memorized and reproduced upon the blackboard when called for, but that they serve as models of logical deductive reasoning; and that the student must see them as such — see that they follow from definitions, axioms, or previously proved propositions. He must not only see the logic of the proof given in the text, but must discover other proofs, if there be such, by making use of other previously studied propositions. He must also see that, had a different definition been given, or some other proposition been known, a different proof might have followed; and thus he should comprehend clearly the complete dependence of each proposition upon the others or upon the fundamental definitions, axioms, and postulates.

But this is not the only end to be sought in geometry. Even though the student may have proved a proposition in various ways, it may never have dawned upon him that each proposition is the natural deduction from some proposition, definition, or axiom already known, and that it might have been discovered by him himself. He is more likely

to feel that the whole subject is entirely foreign to any of his experiences, and that he has performed his whole duty when he has "learned the next two propositions" assigned for each day's lesson. In this country the study of demonstrative geometry is usually put off until the eleventh grade and completed in one year, and is too often a case of "learning the next two propositions."

Dr. Young, in his little book on "The Teaching of Mathematics in Prussia," [1] — a book that should be in the hands of every teacher of secondary mathematics, — gives the following geometry lesson supposed to be given to a class in the first year's geometry, the minimum age for admission to the class being eleven years.

The teacher draws a triangle *ABC* upon the blackboard, and then questions the pupils somewhat as follows, the pupils being called on singly by name in as lively alternation as possible:

How many angles has a triangle ?

Name an angle of the triangle *ABC*. A second. A third.

The teacher draws and defines an *exterior* angle *CAD*.

Who can draw another exterior angle ? (Done repeatedly by various pupils.)

How many exterior angles can be drawn at one vertex of the triangle ?

How many exterior angles can be drawn altogether?

What are the two exterior angles at the same vertex called with regard to each other?

What, therefore, do we know as to the magnitude of these two angles ?

[1] Longmans, Green & Co., New York.

How many exterior angles differing in size can a triangle have at most?

Why is it customary to speak of only one exterior angle at each vertex of a triangle?

In view of this custom, how many exterior triangles would a triangle be said to have?

For convenience, the letter at any vertex shall be used to denote the interior angle and the letter primed the exterior angle at that vertex. The notion of adjacent angles is supposed to have been explained earlier in the course.

What are an interior angle and its adjacent exterior angle called with respect to each other?

What theorem holds for two such angles?

We wish now to compare the magnitude of an exterior angle with that of the two non-adjacent interior angles. For this purpose we regard AB and AC as two non-parallel straight lines cut by a third BA; the last produced forms the angle CAD or A'.

In what positions are A' and B with respect to each other? Likewise A' and C?

Can therefore $A' = B$ or $A' = C$?

To compare the magnitudes of the angles we draw AE parallel to BC and divide A' into the angles CAE and EAD, which angles we call x and y respectively.

Since AE is parallel to BC, there is another angle in the figure equal to y; what is it?

Why are y and B equal? Which are the parallels and which is the secant line?

The teacher marks the equal angles with the same mark.

Is there also an angle in the figure equal to x? What is it?

The teacher also marks these angles equal angles with the same mark, different from that used with the previous pair.

Why are these two angles equal? Which two lines are now the parallels? Which the cutting line?

Since $y = B$ and $x = C$, how large is $x + y$?

To what is therefore the angle A' equal?

What theorem have we thus found?

The proof is now repeated systematically, first with the same figure, then with a different exterior angle of the same triangle, or also with an entirely different triangle, something as follows:

The teacher produces BC beyond C and asks:

What do we assert concerning the exterior angle at C?

What auxiliary line do we draw to facilitate the proof?

What pair of angles are now equal? Why?

(To avoid breaking the main course of thought, the parallels and the secant line are not now called for in detail.)

What follows from the equality?

The author says that some pupil is then called upon to give the entire proof, and at the next lesson the proof is taken up until all the pupils are able to give it in a connected way.

That this kind of teaching is excellent and will create a great interest and give that discipline that the study of geometry should give, cannot be questioned. Several things, however, make everyday teaching of this kind impracticable in our schools as they are at present organized. Our students spend but one year (usually the eleventh grade) on geometry. If they have no texts, they can usually find the one the teacher is using; for but few of our teachers have the time or ability to work out a proper sequence of propositions essentially different from those of the leading texts. With some text, then, those students who want to appear smart will be prepared for the development of each new proposition. Occasional lessons, however, can and

should be given in this way upon the propositions of the advanced lesson.

I think, therefore, that much the better plan is to place a well-written text in the hands of the pupil. Although much that passes for the teaching of geometry when a text is used is merely the training of memory, yet a "complete demonstration" book does not deprive the pupil of all opportunities to think for himself; and when it is properly used such a book is to be preferred to a syllabus of propositions or a "suggestive plan" book.

The Study of Demonstrated Propositions.

Let it not be thought, however, that one is getting geometry when he can give back *verbatim*, in a parrot-like, listless way, the proof of the text.

Lead the pupil to see that in every proposition of plane geometry the *fundamental operation* is either *a comparison of angles, a comparison of lines, a comparison of areas, or a comparison of ratios.*

As each new proposition is studied, let it be classified under its proper head, "a comparison of angles, "a comparison of lines," etc. Let it be understood that these are tools that may be used if we are again to compare angles, lines, or areas.

In taking up the proof in the text-book, I should have the pupils tell the fundamental operation, the chief tool the author used, and other tools that he might have used. I should have them try to dis-

cover why the author used the particular tool he did. Was it to avoid auxiliary lines, or did it make the proof more direct? I should have the pupils give a proof, requiring the use of other tools, and compare this with the proof of the text.

Will this kind of teaching not instill life into a class, and not only make geometry more interesting, but give a grasp of the subject that cannot be obtained from the mere memorizing of a text-book?

The Study of the "Exercises."

While the demonstrated propositions, when taught as suggested above, afford valuable training, they cannot stimulate the spirit of discovery and original investigation as well as the "exercises" found in most of our text-books of to-day.

From the time of its discovery by the Greeks, geometry has been looked upon as furnishing a peculiar and distinct kind of discipline. Plato inscribed above the entrance to his school, "Let no one unacquainted with geometry enter here." To-day if we are asked why the subject is taught in our high schools, we answer glibly that aside from the fact that a certain knowledge of geometry, as that concerning areas and volumes, is useful and should be known, it is given largely because it furnishes almost the only examples of logic in the high-school course; that it teaches the student to consider carefully given data, and to reason from

this to accurate conclusions. Yet whether the student can do this or not depends largely upon how the subject is taught. If the student of geometry is given an exercise that does not involve familiar figures and theorems, although it may be a theorem in some other branch of mathematics, he is usually helpless in discovering a key to the proof. This shows us how much of what we thought in the geometry class was the result of reason must have been the result of a good memory. Suppose the pupil is asked to prove the theorem that "all prime numbers are either of the form $6n+1$ or $6n-1$." If he gets a proof at all, there is generally a lack of definite, systematic method of attacking it, and one is convinced that he is *not* being taught "to consider carefully given data and reason from this to accurate conclusions" to the extent that we might expect.

Geometry can certainly be so taught as to secure to a very satisfactory degree the desired disciplinary value spoken of above; but I fear that too often we are satisfied if the student of geometry can reproduce the book on the blackboard or on paper, and possibly be able to justify each statement of the text by citing some previously studied proposition, axiom, or definition. The excuses one often hears in the geometry class that "I can't remember how it begins," "I have forgotten the next step," or "I don't remember the proposition, but I know how to prove it," lead me to believe that with far too

many pupils and in too many class rooms the subject is more a drill in memory than an exercise in logic. Granting, however, that the student does see every step in the proof, and sees in the proof an example of logical deductive reasoning, the method of proof in our texts is necessarily synthetic, with no hint as to how the proof was discovered or what suggested the theorems. Unless the demonstrated theorems be taken up somewhat in the way that has been briefly suggested, the student is not gaining the power to solve the various propositions that will present themselves in other subjects.

If the student is to get the greatest educational good, discipline in logical deductive reasoning, if he is to acquire a spirit of investigation and discovery, he must not be hurried through the course, learning simply the proofs of the text, but he must supplement this drill by a great deal of original work in various demonstrations and solutions.

The Method of Attack.

Happily the exercises (usually called "originals" in this country and "riders" in England) cannot be grouped into "cases," and methods, rules, or principles given, that enable us to solve all of each class as was the case in our older arithmetics when we came to the subject of percentage. Yet there are

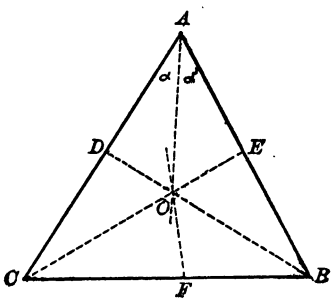
certain principles, or methods of attack, that make the study of an exercise systematic or scientific in the fullest measure; and unless the student is taught to study the exercises in this systematic way — unless he is taught to study the exercises from the standpoint of analysis and thus discover the possible lines of procedure in attacking the demonstration — he is not gaining the power that is going to help him to study the larger problems of life. The student of geometry, when given an exercise, should no more fold his arms and try to recall "some proposition like it," or leaf through other texts hoping to find a proof, than a student of botany or chemistry should analyze a plant or find the ingredients of a compound by searching in his encyclopedia for something that looks like it. He ought to have a definite line of procedure, and be able when a proposition is stated to do something, if it is nothing more than to waste paper, just as the student of chemistry, when asked to analyze a compound, would do something if nothing but have an explosion.

While very few rules can be laid down, these few give system to our work. Let us notice the general lines of procedure.

1. First, *assume that the theorem is true and draw an accurate figure*, and the accurately drawn figure will often suggest a proof. For example, certain triangles may appear to be congruent; if they are, this will lead to the discovery of some other

relations that make the proof evident; hence we seek to prove these triangles congruent.

By assuming the theorem to be true is meant that if, for example, certain lines or angles are to be proved equal, they are to be drawn so. Drawing an accurate figure means not only that if lines are equal, parallel, or perpendicular, they must be drawn so; but also that lines are *not* to be drawn equal, parallel, or perpendicular unless they are given so. A triangle is not to be drawn with two sides equal unless the theorem calls for an isosceles triangle, nor a quadrilateral drawn with any two sides equal or parallel unless the exercise calls for a special quadrilateral of that kind. An inaccurate figure may seem to show relations that do not exist, as in the following example taken from Ball's "Mathematical Recreations," in which all triangles are proved to be isosceles.



Given the Δ ABC, any triangle.

To prove that ABC is isosceles.

Proof. 1. Draw $FO \perp$ bisector of BC.

2. Let AO bisect $\angle A$, and intersect OF at O.

3. Draw OD and $OE \perp$ respectively to AC and AB.

4. Also draw OC and OB.

5. Now $\because \angle \alpha = \angle \alpha'$ (def. of bisector) and $OA = OA$.

6. $\therefore$ *rt* $\triangle DOA \cong$ *rt* $\triangle OEA$.

7. $\therefore$ $AD = AE$.

8. Also $OD = OE$.

9. $\because$ $CF = FB$ (given) and $OF = OF$ $\therefore$ *rt* $\triangle CFO \cong$ *rt* $\triangle FBO$.

10. $\therefore$ $OC = OB$.

11. $\therefore$ *rt* $\triangle COD \cong$ *rt* $\triangle OBE$.

12. $\therefore$ $CD = BE$.

13. $\therefore$ $AD + DE = AE + EB$ or $AC = AB$ and the $\triangle$ is isosceles.

Or,

Suppose OA and OF do not meet.

1. Then $OA \parallel OF$ and $\therefore$ $OA \perp CB$.

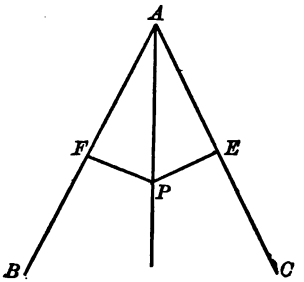
2. But $\because$ OA bisects $\angle A$, we would then have two right $\blacktriangle$ with acute $\angle \alpha = \angle \alpha'$ and a leg common, therefore congruent.

3. $\therefore$ $AC = AB$ and the $\triangle$ is isosceles.

If the figure had been accurately drawn, OF and OA would not have met within the triangle.

2. After the figure has been drawn, *get clearly in mind just what is given in the figure and just what is to be proved; i.e.* the *data* and the *conclusion.* It is also very important that *the student know the definition of all terms in the theorem.* For example, suppose he has the theorem that,

All points on the bisector of an angle are equally distant from the sides.



Now instead of the *hypothesis* and *conclusion* of the theorem the student should follow Pascal's advice and "*substitute the definition for the name of the thing defined*" and say,

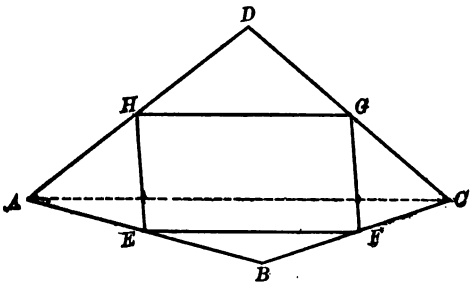
Given angle FAP equal to angle PAE, and PF perpendicular to AB, and PE perpendicular to AC.

To prove that $PF = PE$.

3. Next, *recall all the propositions that can have a bearing upon the exercise under consideration.* I believe that a great part of the difficulty that a beginning student has comes from his not having clearly in mind all these fundamental facts that he needs, and in not having them classified so that he can use them. For example, if lines are to be proved equal, the student must know all the propositions that refer to equal lines, such as the corresponding sides of congruent triangles, opposite sides of a parallelogram, etc. This in

turn necessitates his knowing all the theorems concerning triangles, parallelograms, etc., and proving the triangles congruent will necessitate his proving angles equal, and this likewise requires the theorems concerning equal angles. Now, to get these fundamental facts fixed, I should require the student, when first taking up the study of original exercises, to recall with each exercise *all* the propositions that might possibly suggest a proof. This review will fix firmly all the fundamental theorems in his mind. As an example of this third suggestion, suppose we have the theorem that

The figure formed by joining the middle points, in order, if any quadrilateral is a parallelogram.



Given the quadrilateral *ABCD*, and *EFGH* a quadrilateral formed by joining the middle points of *AB*, *BC*, etc.

To prove that *EFGH* is a parallelogram.

Analysis. When is a quadrilateral a parallelogram ?

(1) opposite sides parallel (def.).

(2) opposite sides equal.

(3) opposite angles equal.

(4) two sides equal and parallel.

(5) the diagonals bisect each other.

Suppose (1) is chosen, how are lines proved parallel ?

(*a*) alternate-interior angles equal, corresponding angles equal, etc.

(*b*) the interior angles on one side of a transversal supplementary.

(*c*) a line bisecting the sides of a triangle is parallel to the base.

Now (*a*) and (*b*) will require all the theorems concerning equal angles, as congruent triangles, and parallel lines. In the same way (2), (3), (4), and (5) should be taken up.

The student is now ready, after following each suggestion to its limit, to select the tools which it is *possible* to use, as well as those which it is *best* to use. When he has done this for some time, until he has "at his tongue's end" all propositions and corollaries that he has already proved, he sees in each of the terms involved in a theorem not only the *one* definition of the term, but *many*, and is able to select the particular ones that will lead to a proof. For example, he comes to think of parallel lines not only as those that do not meet, but also as those that make with a transversal equal alternate interior angles, equal corresponding angles, or the two interior angles on the same side of the transversal supplementary. He thinks of a parallelogram not only as a quadrilateral whose opposite sides are parallel, but also as a quadrilateral whose

opposite sides are equal, two of whose sides are equal and parallel, whose diagonals bisect each other, etc. He is now in a fair way not only to succeed in geometry, but is gaining that spirit of self-reliance and that knowledge of his own power to do things for himself that will help him in all his work.

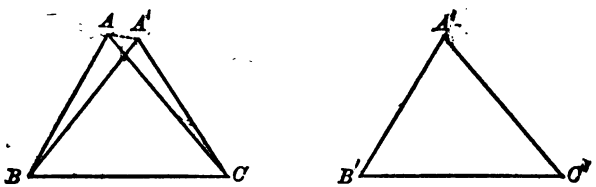
4. Having recalled all propositions that can possibly relate to the thing to be proved, certain ones may be seen to apply, while again none of these may come directly under the figure as it is given, as could have been shown to be the case with the figure just studied. Hence *now, and not until now, is the student ready to draw, and draw intelligently, the auxiliary lines needed to make some of these theorems apply.* For example, questions (4) and (*c*) in the analysis above suggest the auxiliary line *AC* or *BD*.

5. *Now supposing that we have discovered theorems that may be made to apply, or by drawing auxiliary lines they may be made to do so; we reverse the steps of the analysis and give the regular synthetic proof.*

6. After carefully analyzing the figure as suggested above, if no direct proof can readily be discovered, *then assume the theorem false* and by analysis prove that the assumption leads to an absurdity. This method is called the *indirect method* or *reductio ad absurdum*. As an example of this, take the proposition that,

Two triangles are congruent when the three sides of the one are equal respectively to the three sides of the other.

In the other two cases of congruent triangles that text-books generally give before this one is taken up, the triangles have angles in each equal, so that, when certain sides are made to coincide, it is known where the other sides *must* fall. In this case, however, since nothing is known of the angles, we do not know, when equal sides are made to coincide, where the other sides are going to fall; hence we may assume that they do not coincide, but fall as in the figure below, and then prove the assumption absurd.



Proof. 1. Place $\triangle A'B'C$ upon $\triangle ABC$ so that $B'C$ will coincide with BC, and suppose that A' does not fall upon A.

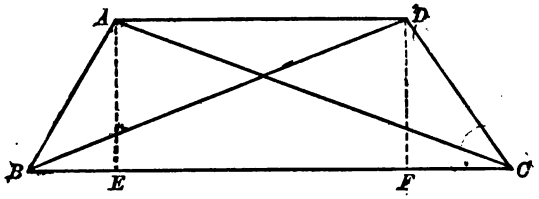
2. $\because A'B' = AB$, and $A'C = AC$, $\therefore$ $\triangle A'BA$ and $A'CA$ are isosceles.

3. $\therefore$ the perpendicular bisector of $A'A$ will pass through both B and C, which is absurd.

4. $\therefore$ the assumption that A' does not fall upon A is absurd and the $\triangledown$ must coincide throughout, and hence are congruent.

To illustrate in a single exercise the six steps that have been discussed, suppose we use the theorem that,

If the diagonals of a trapezoid are equal, the trapezoid is isosceles.



1. ∵ the trapezoid is to be proved isosceles it should be drawn so.

2. **Given** the trapezoid $ABCD$ with $AC = BD$.

3. **To prove** that $ABCD$ is isosceles. Now, use Pascal's advice and "substitute the definition in place of the name of the thing defined," and say, *to prove* $AB = CD$.

4. **Analysis.** To prove lines equal requires,
   1. congruent triangles,
   2. parallelograms, etc.

(*a*) To prove the triangles congruent requires that we have,
   1. the three sides of one equal, respectively, to the three sides of the other.
   2. two angles and a side of one equal, respectively, to two angles and a side of the other, or
   3. an angle and two sides of one equal, respectively, to an angle and two sides of the other (except in the ambiguous case).

Now the first suggestion under (*a*) is seen to be impossible, and to use the second or the third, we must have equal angles; and to prove the angles equal requires:

1. congruent triangles,
2. vertical angles,
3. parallel lines.

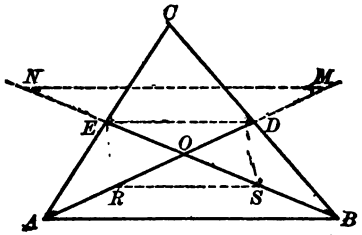
(*b*) Or if our tool be that of parallelograms; to prove a figure a parallelogram requires:

1. opposite sides parallel,
2. a pair of opposite sides parallel and equal,
3. opposite angles equal, etc.

Now, observing our figure, we see that none of these theorems can be used as the figure stands; we see therefore the need of other lines, and seek to draw some that will give a figure or figures that will enable us to use some of the theorems recalled. Suppose our proof to be that of congruent triangles. To get the necessary data in order to prove triangle *ABC* congruent to triangle *BCD*, we must have in addition to what we already have, angle *ACB* equal to angle *CBD*; hence we form congruent triangles involving angles *ACB* and *CBD*. To do this we know that *two right triangles are congruent when the hypothenuse and a leg of one equal the hypothenuse and a leg of the other*, and that *parallels intercepted between parallels are equal;* also that *perpendiculars to the same line are parallel;* hence we drop from *A* and *D* perpendiculars to *BC*, and prove triangles *AEC* and *BFD* congruent. Now we have angle *ACB* equal to *BCD*, and can prove triangles *ABC* and *BCD* congruent, and hence the proposition.

Or suppose the theorem is that,

The medians of a triangle meet in the point of trisection farthest from the vertex.



1. Draw an accurate figure. Now we have

2. **Given** triangle ABC and medians AD and BE.

3. **To prove** (1) that AD and BE intersect at some point O. *i.e.* prove that AD and BE are not parallel.

(2) that O is a point of trisection of AD and BE. This means either that

$$OD = \tfrac{1}{2}\, OA \text{ and } OE = \tfrac{1}{2}\, OB,$$

or that $$OA = 2\, OD \text{ and } OB = OE.$$

4. **Analysis.** — Since no theorems have been given dealing with such relations as one segment half or double another, this theorem can only be proved by bisecting OA at R and OB at S, or doubling OD and OE by extending OD to M and OE to N, and we now have either of two new conclusions, namely:

To prove (1) that $OA = OM$ and $OB = ON$,

or (2) that $OD = OR$ and $OS = OE$.

Now if we analyze each of these as we did in the first example, a number of proofs can be discovered.

Then summarizing the steps that should be taken in dealing with theorems, we may say:

1. Assume the theorem true and draw an accurate figure.

2. Get clearly in mind the hypothesis and the conclusion, being sure to understand the full meaning of all terms involved, substituting when necessary "the definition in place of the name of the thing defined." In other words, discover the fundamental operation to be performed, which, in any theorem of plane geometry, is a comparison either of angles, lines, areas, or ratios.

3. Recall all theorems or definitions that refer to the thing to be proved; the equality of angles, lines, areas, etc.

4. If none of these seem to apply to the figure as it stands, try to draw auxiliary lines that will involve the elements wanted, and that will also give a figure to which some of the known theorems will apply.

5. If successful, then reverse the process, and give the regular synthetic proof.

6. But if a *direct proof* is difficult to find, then *assume the theorem false*, and show that such an assumption leads to an absurdity. It might also be well to call attention to the fact that a *converse proposition* is generally more easily proved by the *indirect method*, or *reductio ad absurdum*.

The Solution of Problems.

The solution of a problem, together with the discussion of

(1) the number of solutions in general;

(2) the relations existing in the data that give a definite number of solutions;

(3) or an indefinite number of solutions; and

(4) that make the solution impossible,

is an excellent supplement to the discipline obtained from the demonstrations of theorems. Such discipline ought to give strength and self-reliance that will better prepare one to solve the various problems that may arise in other subjects, for the manner of approaching the solution of a problem is the same in all subjects: *it is approached through analysis.*

1. In seeking a key to the solution of a geometrical problem, in order to aid the analysis, it is generally best to *assume the solution performed and from the elements of the figure recall some known relations that have already been proved.* Having discovered enough of these relations, make the construction depend upon them.

2. In beginning the study of problems already solved in the texts, the student should be made to know at once that *all solutions must depend upon some known theorem or theorems,* and that these should be recalled. I should have the student recall other theorems, if such exist, that might suggest a

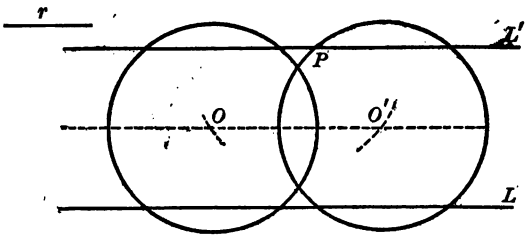
solution other than the one given in the text. The student must see not only that to solve a problem he must be able to recall some known proposition that makes the construction evident, but he must see also that all problems must be reduced to one or both of two fundamental problems:

I. *to draw a straight line between two points*, or

II. *to describe a circle of a given radius about a given point as center.* He must see, moreover, that the required points are found by the intersection of two lines, straight or curved; and hence that almost every solution must depend upon *the intersection of certain loci.*

3. The simplest and most common problems of elementary geometry are those in which the analysis leads to the discovery that the points wanted are on certain loci, hence at their intersection. This method is called **the intersection of loci.** To illustrate:

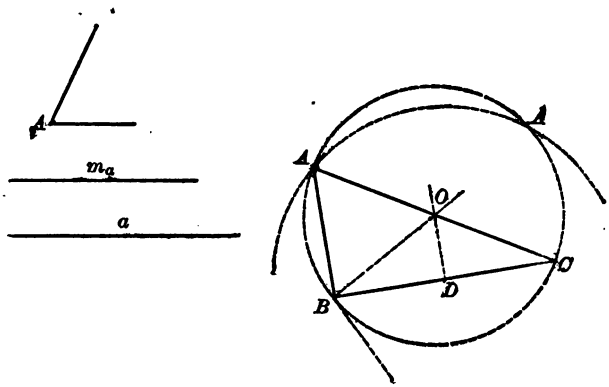
To describe a circle of given radius to pass through a given point and cut off equal segments from two parallel lines.



Now, this depends upon finding the center of the circle — *a point.* The analysis leads to the discovery that the center of this required circle is the intersection of *the locus of points equidistant from two parallel lines*, and of *the locus of all points equidistant from the given point;* that is, of *a straight line* and *a circumference.* Since, in general, a straight line will cut a circumference in two points, two solutions are possible. The student should also discuss the other cases, when but one solution is possible, and when the solution is impossible.

Or suppose we are

To construct a triangle having given a side, the angle opposite, and the median to that side.



Having a side given, the problem is reduced to finding the third vertex — *a point.*

Now our analysis leads to the discovery that the required point is on the locus of points from which the given line BC subtends the given angle A, and also that it is on the locus of points at a given distance m_a from a given point D, the mid point of BC. Knowing what these loci are, the solution is evident.

The method of *intersection of loci* is used to such an extent in constructive geometry that the student ought to be made familiar with the common theorems of loci, and have them fresh in mind when taking up this subject.

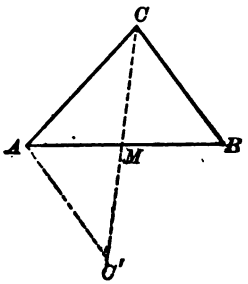
While I am aware that most students go through some sort of analysis without perhaps being even conscious of it, I am sure that the most good will come from the course if the student is made to realize that there is a systematic analysis involved in every discovery of a solution, and if he is required to begin all solutions either written or oral by giving the analysis that led him to see the solution.

4. It will often occur that the analysis of a figure will not reveal any known relations that will give the means of determining the required points without drawing auxiliary lines that will transform the given figure into a new one involving certain elements of the old, just as was done in the case of theorems. This might be called the **method of transformation**. Just as the most difficult theorems were those that had to be transformed by

auxiliary lines, so will this class of problems give the most trouble, and the skill to see the needed auxiliary lines will largely determine a student's success with exercises of this sort.

As an example, suppose we are

To construct a triangle having given two sides and the median to the third side.



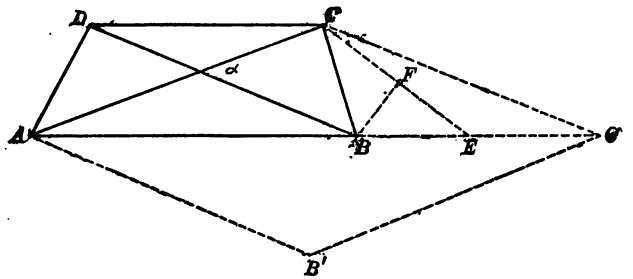
Now the simple figure with the median drawn gives no way of determining the points A and B when C is chosen; but, by continuing the median CM to C', making $MC' = MC$, the solution becomes evident, and depends upon constructing a triangle whose sides are the two given sides and twice the median. This construction gives a second vertex, A, from which B can be found by drawing AB through M, making $MB = AM$.

5. The auxiliary lines needed to transform the figure into a new one, however, are usually drawn parallel to lines in the original figure, *i.e.* certain lines are considered to be moved parallel to them-

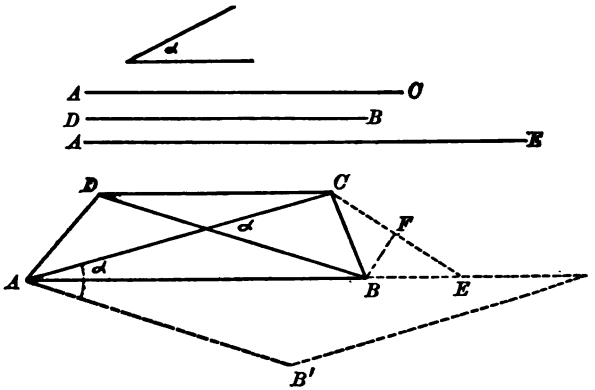
selves, called translated, thus giving a new figure which involves elements of the old and which makes the solution evident. Such a method of solution is called the **method of parallel translation.**

As an example, suppose we are

To construct a trapezoid having given the diagonals, the angle made by them, and the sum of two adjacent sides.



Assuming the problem solved and the trapezoid to be $ABCD$, we get no relations from the figure as it stands that show where to intersect the diagonals or where to draw a side if a diagonal is fixed; hence we must transform the figure by auxiliary lines. Drawing through A, AB' parallel and equal to DB, and through B', $B'C'$ parallel and equal to AC, it can easily be proved that the side AB of the trapezoid coincides with the diagonal of the parallelogram thus formed, and the solution is evidently as follows:



1. Mark off on any straight line a length equal to AC.

2. Construct with A as vertex and with AC as one side, an angle equal to $\angle \alpha$, and let the second side be AB', equal to DB.

3. Construct the $\square AB'CC$.

4. Draw diagonal AC'.

5. Mark off AE equal to the sum of the two adjacent sides.

6. Connect C and E.

7. Erect FB the perpendicular bisector of CE cutting AC in B.

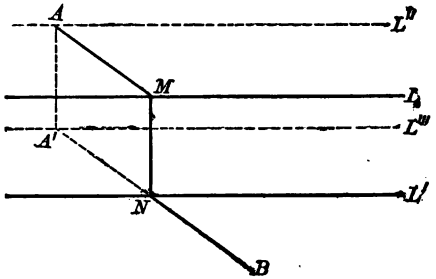
8. Draw $BD \parallel AB'$ and equal to BD, and connect the four vertices A, B, C, and D, and the trapezoid will be the one required.

Now a proof and a general discussion of the special cases should follow.

A more simple illustration of parallel translation is given in the problem,

To join A and B, two points, exterior to two

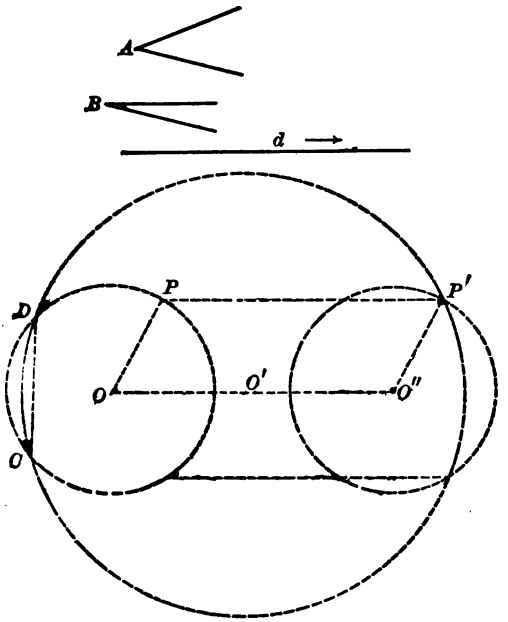
parallels L and L', and on opposite sides of them, by the shortest broken line which shall have the intercept between the parallels perpendicular to the parallels.



To solve the problem one of the intersections with the parallels must be found. Now if the parallels L and L' be translated parallel to their original position to the positions L'' and L''', keeping their original distance apart, it is evident that $AA'B$ is the shortest distance between A and B, AA' being perpendicular to L'' and L'''. It is also evident that N, where $A'B$ intersects L', must be the point where the required line intersects L' and the solution is now seen.

Another interesting example of *parallel translation* is found in the following problem:

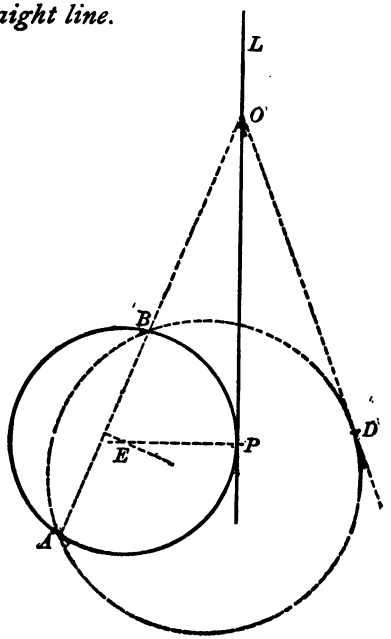
From a ship two known points are seen under a given angle; the ship sails a given distance in a given direction, and now the same two points are seen under another known angle. Find the two positions of the ship.



Analysis. — Let the given points be C and D, the given distance and direction be represented by d, and the given angles be A and B. It is evident that one position of the ship must be on the arc of a segment of circle O, constructed on chord CD, and containing angle A; and that the other position is on the arc of the segment of circle O', constructed on chord CD, and containing the angle B. Now since the position on circle O is on circle O' after traveling the distance d in the given direction, it is evident that if we translate the circle O this distance and in this direction the intersection of the two will give the second position of the ship, and the point of intersection on circle O, when translated back to the first position, will be the other position; hence P and P' are the positions sought.

6. While nearly all problems are finally solved by the first method given, *i.e.* by the method of *intersection of loci*, still it often occurs that before one finds the required loci a difficult analysis has to be performed, and then the method of attack is called simply the **method of analysis**, which, in reality, is the fundamental method involved in the search for a key to the solution of any problem. As a problem of this kind, let us suppose that we are

To draw a circle whose circumference shall pass through two given points and also be tangent to a given straight line.



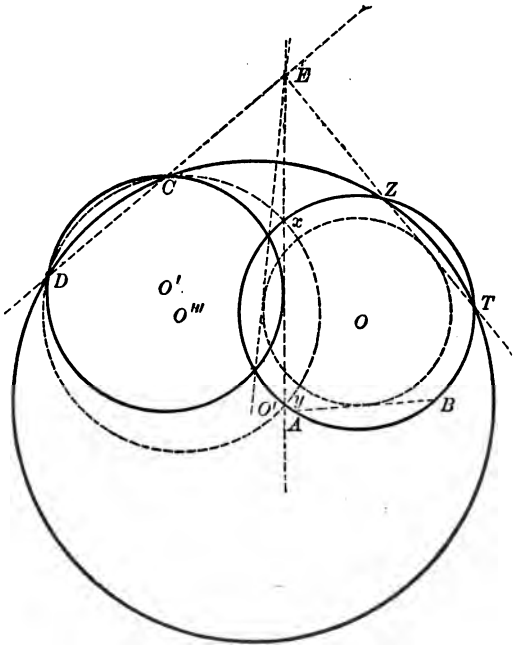
Suppose A and B to be the given points, and L the given line. Now if A and B be joined and this line extended to meet L in O, it is known that the square of the segment on L from O to the point of tangency of the required circle is equal to the product of the segments OA and OB; but it is known that this is also true of the length of the tangent from O to *any* circle through A and B; hence one has simply to describe any circle through A and B and draw from O a tangent OD to this circle, and then OP on L, and equal to OD, will give the point of tangency P on the given line, and the center of the required circle is easily determined.

Again, to "assume the problem solved" will not always give a figure that will lead to the discovery of the solution, nor can the proper auxiliary lines always be readily seen. In a case of this kind, one can, without a figure, simply recall theorems that relate to the thing wanted. For example,

To describe a circle passing through two given points and cutting off from a given circle an arc of given length.

In this case a figure does not materially help us. We know without a figure that we must find two points on the given circle and a given distance apart, that are concyclic with the two given points. We recall the theorems referring to concyclic points. If we recall the theorem that, *if two points are taken on each of two rays of a pencil such that the rectangle contained by the segments of one from the vertex is equal to the rectangle contained by the segments of the other, the four points are concyclic;* and the related one that, *if a pencil of rays cuts a circum-*

ference, the rectangle contained by the segments from the vertex is constant whatever ray may be taken, together with certain more elementary problems, the solution becomes evident, and is as follows:



1. Let O be the circle and C and D the points, and AB the arc.

2. Draw through DC any circle cutting the given circle O.

3. Draw the secant DC and the common secant to the two circles and let them meet at E.

4. Now it is evident that any secant of O through E will cut O in points concyclic with C and D.

5. But the required secant must cut the circle O so that the intersected arc will equal the arc AB, hence the required secant must be tangent to a circle concentric with O and tangent to chord AB, and since from point E two tangents can be drawn to this circle, two solutions are possible.

Since no two problems are solved alike, the power to attack an exercise successfully comes from a great deal of practice, coupled with close observation of the method by which each exercise was studied, rather than from the study of the few methods given here; but I believe that if a student has had his attention called to these general methods, he will become much stronger and more self-reliant, and be able, to a larger degree, to handle the problems of any other department of school instruction or the problems that come up in daily life.

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